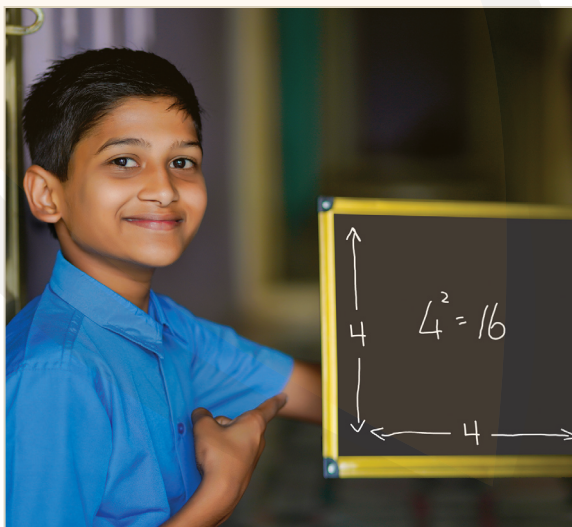


# Squares of Numbers and of Polynomial Expressions

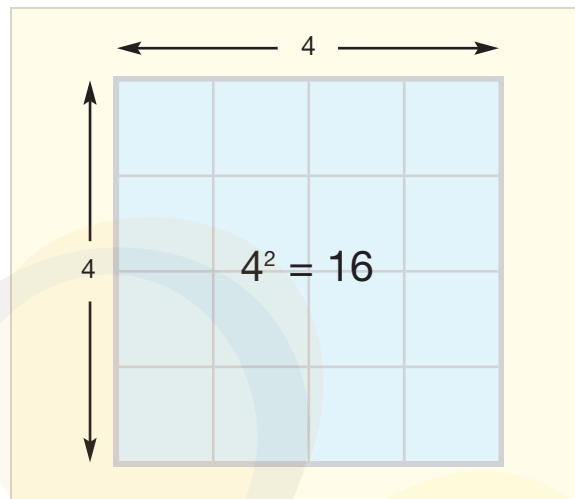
## Background Information



The square of a number is made when two equal factors are multiplied together

## Squares of Numbers

The square of a number is the product that results when two equal factors are multiplied together. For example,  $4 \times 4$  is the square of 4. In this example, the two equal factors 4 and 4 are multiplied together. Since  $4 \times 4 = 16$ , the square of 4 is 16. In other words, four squared equals sixteen, which can be written thus:  $4^2 = 16$ . The small 2 in  $4^2$  is called an exponent. The exponent tells how many equal factors are to be multiplied together. The 4 in  $4^2$  is called the **base**. A base in exponential notation is the number that is taken as an equal factor as many times as the exponent specifies.



The square of 4 is 16

Another way to talk about exponents is to call them **powers**. A power of a number is the product obtained when a base is used as a repeated factor 0 or more times. In the above example, 16 is the second power of 4, because 16 is the product obtained when 4 is used as a repeated factor twice. Said otherwise, 4 to the power of 2 equals 16.

Students first encounter the concept of squares and squaring by building the Decanomial Square, which is a physical representation of all the products from  $1 \times 1$  to  $10 \times 10$ . Because the Decanomial Square includes the squares from  $1^2$  through  $10^2$ , students have an opportunity to see where these squares fit into the physical arrangement of products.

## Polynomial Expressions

Students must understand the concept of polynomial expressions (particularly binomials and trinomials) for their work with squares and square roots.

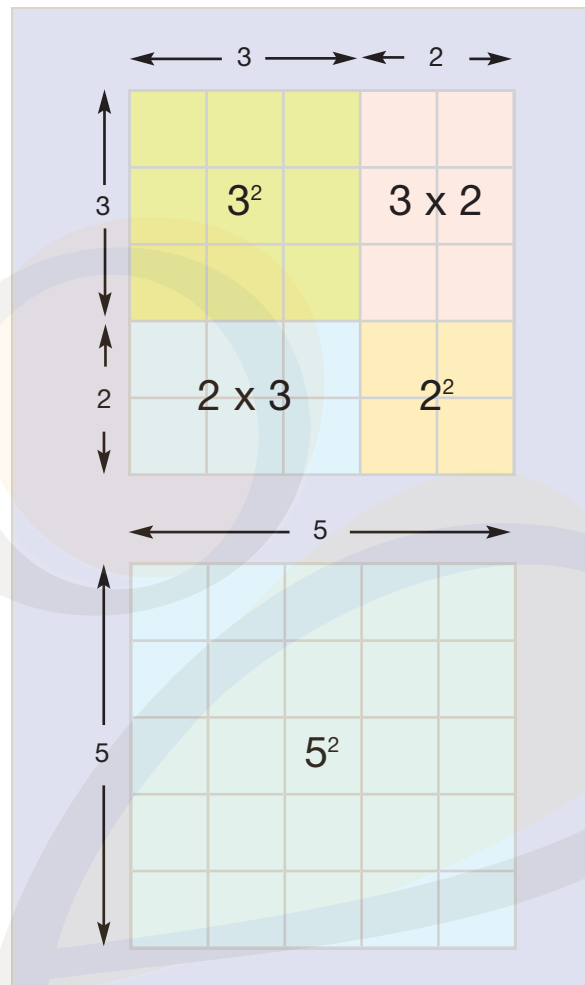
Mathematical expressions generally occur as part of a mathematical sentence such as an equation or inequality. For example, the expression  $a + 3$  is part of this equation:

$a + 3 = 5$ . An expression combines one or more terms using addition, subtraction, or both. Expressions may consist of numbers only, numbers plus symbols, or symbols only. An expression containing only numbers is called a **numerical expression**. An expression that includes at least one symbol is called an **algebraic expression**. The following are examples of algebraic expressions:  $a$ ;  $2a - 6$ ;  $a - 2a + 4$ .

An expression may have one, two, three, or more terms. Expressions have different names depending on how many terms they have:

- A **monomial expression**, also called a monomial, is an expression with just one term. Examples:  $3$ ;  $a$ ;  $7z$ .
- A **binomial expression**, also called a binomial, is an expression with two terms. Examples:  $3 + 4$ ;  $a + b$ ;  $2x - 6$ .
- A **trinomial expression**, also called a trinomial, is an expression with three terms. Examples:  $3 + 4 - 5$ ;  $a - 2b + 4$ ;  $2x - y + 6z$ .
- A **polynomial expression**, also called a polynomial, is simply an expression with one or more terms. Polynomials include monomials, binomials, trinomials, and expressions with four or more terms.

Any monomial can be described as a binomial or a trinomial. For example, 9 equals the binomial  $4 + 5$ . It also equals the trinomial  $2 + 3 + 4$ . Students apply this concept when using the material to find squares and square roots.



The square of  $(3 + 2)$  equals the square of 5

## Squares of Binomials

Binomials can be squared just like any other number. For example, the square of the binomial  $(3 + 2)$  is  $(3 + 2)^2$ . The square  $(3 + 2)^2$  can be written as follows:

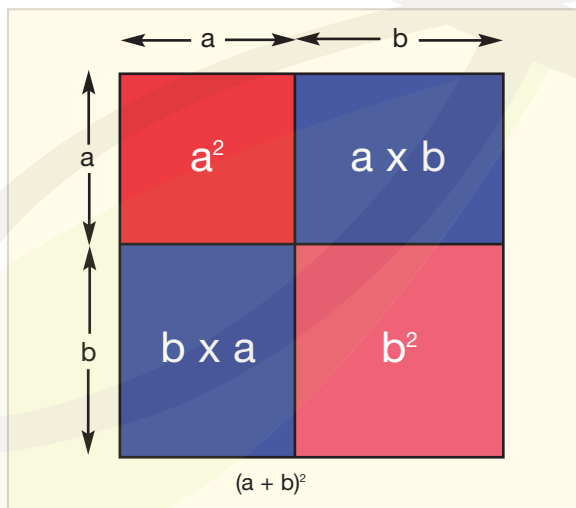
$$(3 + 2) \times (3 + 2)$$

The value of this square can be found by expanding the product. **Expanding** an expression means writing it without parentheses. To expand this product, it is necessary to use the **distributive property of multiplication**, which states that multiplication can be distributed over addition or subtraction. First, the 3 in the first set of parentheses is multiplied by both

terms in the second set of parentheses to obtain  $(3 \times 3) + (3 \times 2)$ . Then the 2 in the first set of parentheses is multiplied by both terms in the second set of parentheses to obtain  $(2 \times 3) + (2 \times 2)$ . The contents of each set of parentheses are simplified by multiplying, and the resulting expression is simplified by adding the four terms:

$$\begin{aligned} &(3 \times 3) + (3 \times 2) + (2 \times 3) + (2 \times 2) \\ &= 9 + 6 + 6 + 4 \\ &= 25 \end{aligned}$$

This result shows that the square of the binomial  $(3 + 2)$  is 25, just like the square of the equivalent monomial 5. The fact that the two squares are equal may be seen on the accompanying illustration. Note that the square of  $(3+2)$  in the illustration is divided into four parts, like the Binomial Guide Square (illustrated in the Introduction to this section).



The square of the binomial  $(a + b)$

Once students are experienced at building the squares of numerical binomials using the material, they come to see that squaring a binomial follows the same pattern no matter what the terms of the binomial are.

In fact, they see that they can represent the square of any binomial as  $(a + b)^2$ . This leads the students into the algebraic equation for the square of a binomial, as follows. The distributive property of multiplication means that each term in the first set of parentheses is multiplied by both terms in the second set of parentheses:

$$\begin{aligned} &(a + b)^2 \\ &= (a + b) \times (a + b) \\ &= a \times (a + b) + b \times (a + b) \\ &= a^2 + ab + ba + b^2 \\ &= a^2 + 2ab + b^2 \end{aligned}$$

The square of the binomial  $(a + b)$  is shown in the accompanying illustration. In the illustration, the largest square represents  $(a + b)^2$ . The four interior parts of the largest square represent the four terms of the algebraic equation,  $a^2 + ab + ba + b^2$ .

### Grouping Like Terms

The square of an algebraic binomial contains four terms:  $a^2 + ab + ba + b^2$ . However, the commutative property of multiplication states that  $ab = ba$ , so in fact the above can be expressed as  $a^2 + ab + ab + b^2$ , or simply as  $a^2 + 2ab + b^2$ . Putting together terms that are made up of the same symbols is called **grouping like terms**. The terms are expressed with the letters in alphabetical order, so  $2ab$  rather than  $2ba$ .

## ACTIVITY 7

# Finding the Algebraic Expression for a Binomial Square Greater Than 10



Finding the algebraic expression for a binomial square

### Purpose

To understand the algebraic expression for the square of a binomial greater than 10.

### Material

Algebraic Pegboard Material.

Illustration titled Place values and their colors (see NAMC's Curriculum Support Material).

Table of squares (see NAMC's Curriculum Support Material).

Monomial Guide Square.

Binomial Guide Square.

Graph paper.

Coloring pencils.

Whiteboard and whiteboard markers.

White slips of paper.

Felt markers.

Math journals and pencils.

### Presentation

- Most Montessori teachers present this concept in Year 6.
- Invite a small group of students to a mat or table to learn how to write the algebraic expression for the square of a binomial greater than 10.

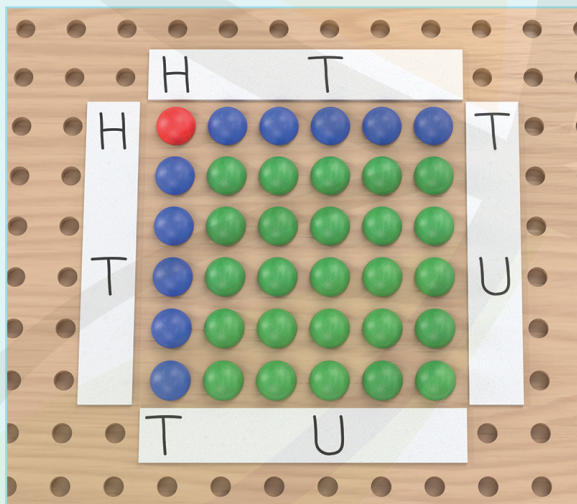
### Reviewing the Binomial Square with Numerical Values

- Begin with the hierarchical square of 15 created in the previous activity. Invite the students to use the white slips of paper to make labels representing the values of the bottom and left sides. The label for each side will be  $(10 + 5)$ . Point out that this is a binomial made from the tens value and units value of each side.
- Given that the binomial  $(10 + 5)$  represents the value of the bottom and right sides of the square, the total value of the square can be found by multiplying  $(10 + 5) \times (10 + 5)$ , which equals  $(10 + 5)^2$ . In other words, the hierarchical square represents the square of a binomial whose value is greater than 10. With the students, calculate the value of the binomial  $(10 + 5)$  mathematically:

$$\begin{aligned}
 &(10 + 5)^2 \\
 &= (10 + 5) \times (10 + 5) \\
 &= [10 \times (10 + 5)] + [5 \times (10 + 5)] \\
 &= [(10 \times 10) + (10 \times 5)] + [(5 \times 10) + (5 \times 5)] \\
 &= 100 + 50 + 50 + 25 \\
 &= 225
 \end{aligned}$$

### Expressing a Binomial Square Using Letters Instead of Numbers

- Invite the students to use white slips of paper to label all four sides of the square in terms of the values of the pegs using the following abbreviations: H = hundred, T = ten, and U = units.

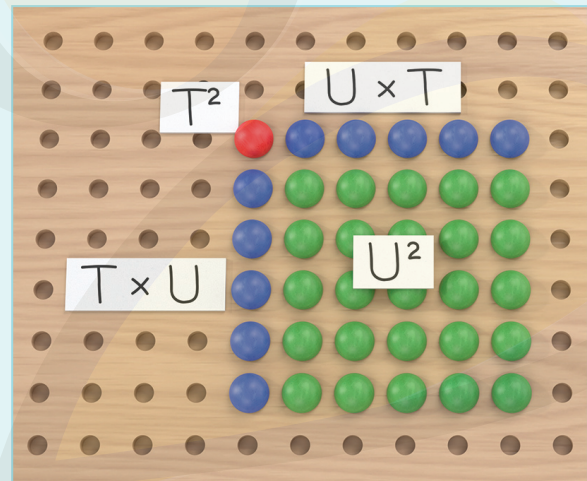


The hierarchical values of the sides of the  $(10 + 5)$  binomial square

- Invite the students to say if they have ever seen this kind of binomial before, with letters in it. Confirm and discuss that the letters can be used as variables. Add that because the pegs have set hierarchical values, the H can only represent the values 100, 200, 300, and so on up to 900, the T can only represent

the values 10, 20, 30, and so on up to 90, and the U can only represent the values 1, 2, 3, and so on up to 9.

- Invite the students to make labels that represent the interior of the square in terms of the values of the pegs shown on the bottom and right sides. In this case, the labels will be  $T \times T$ ,  $T \times U$ ,  $U \times T$ , and  $U \times U$ .



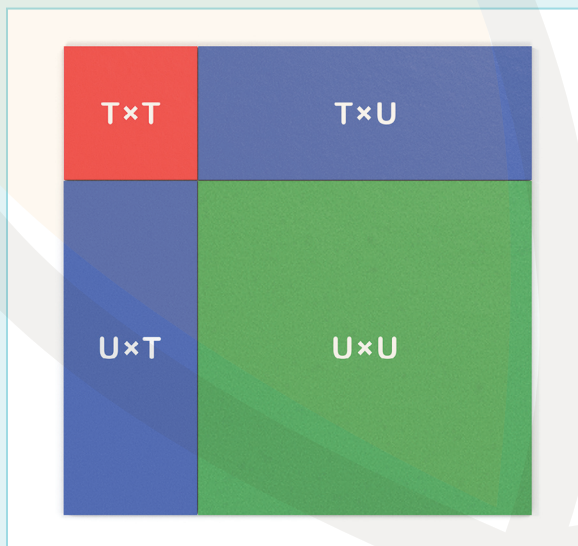
The hierarchical values of the interior of the  $(10 + 5)$  binomial square

- Invite the students to say whether they have seen this pattern before. Confirm and discuss that they are the same results produced by squaring any binomial. That is, if T and U are taken as variables, they can be used to create the binomial equation:

$$\begin{aligned}
 &(T + U)^2 \\
 &= (T + U) \times (T + U) \\
 &= [T \times (T + U)] + [U \times (T + U)] \\
 &= [(T \times T) + (T \times U)] + [(U \times T) + (U \times U)] \\
 &= T^2 + 2(TU) + U^2
 \end{aligned}$$

## Relating the Binomial Guide Square to Squares on the Algebraic Pegboard

- Place the Binomial Guide Square in the work area. Point out that this square represents the arrangement of the pegs on the Algebraic Pegboard. Place the Monomial Guide Square in the lower right corner of the Binomial Guide Square to provide a  $U \times U$  label.



The Binomial Guide Square with the Monomial Guide Square show the same arrangement of elements as the Algebraic Pegboard

- Explain that the Binomial Guide Square represents the values in a hierarchical binomial square built from a binomial of tens and units.
- Explain that this representation of a binomial square sets limits on the values that can be represented by the binomial square precisely because it is built only from 10s and units. With the students, build the smallest possible binomial square within these limitations. Discuss and clarify that the smallest square is built by squaring the smallest possible binomial made of 10s and units:  $(10 + 1)$ .
- Invite a student to follow the Binomial Guide Square to complete the hierarchical square. The square consists of a single 100s peg with a 10s peg below and to the right, and a unit peg in the remaining corner. From this square, it can be determined that  $(10 + 1)^2 = 121$ .
- Invite the students to consider what values are produced in the interior of the smallest hierarchical binomial square. Invite the students to say what the  $T \times T$  square is built from and why. Confirm it is made of 100s pegs because any value in the 10s pegs times any other value in the 10s pegs will always give a value in the 100s. Confirm this by showing that the smallest 10s peg (10) times itself gives 100. Use the same process to show that the other parts of the interior consist of 10s ( $T \times U$ ,  $U \times T$ ) and units ( $U \times U$ ).
- With the students, build the largest possible binomial square within the limitations imposed by the Binomial Guide Square. The largest square is made by squaring the largest possible binomial built from 10s and units:  $(90 + 9)$ . The square consists of a  $9 \times 9$  square of 100s pegs in the upper left,  $9 \times 9$  squares of 10s pegs in the upper right and lower left, and a  $9 \times 9$  square of unit pegs in the lower right corner. From this square, it can be determined that  $(90 + 9)^2 = 9,801$ . Discuss and clarify that the values of the sides and the interior of the square match the Binomial Guide Square:  $90^2 + 2(90 \times 9) + 9^2$ .

- As with the smallest hierarchical binomial square, invite the students to consider what values are produced in the interior of the largest binomial square. Confirm that they remain 100s, 10s, and units in the same arrangement as the smallest square.
- Ask the students to use their journals and graph paper to describe what the Binomial Guide Square represents, including its limitations, and to draw and color a binomial square.

## Extensions

- Beginning with binomials of values between 10 and 100, build hierarchical binomial squares using squares from the Cubing Material, together with Colored Counting Bars. Use the illustration titled Place values and their colors to interpret the values of the colors.
- Using the illustration titled Place values and their colors as a guide, begin with two squares from the Cubing Material, one whose total value is 100 or more (for example,  $10^2$ ,  $20^2$ , etc.) and one whose total value is less than 100 (for example,  $1^2$ ,  $2^2$ , etc.). Place them corner to corner diagonally, and then try to write down what will be needed to complete the square of the binomial. Fill in the square with Colored Counting Bars to check answers.